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Recognizing weakly convex visible polygons

Supriya Biswas [a,1], D. Chithra Prasad [b], Sudebkumar Prasant Pal [a,*,2]

[a] *Department of Computer Science and Engineering, Indian Institute of Technology, Kharagpur-721302, India*

[b] *T.K.M. College of Engineering, Quilon, Kerala, India*

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Abstract

Two points inside a simple polygon are said to be *convex visible* if the Euclidean shortest path between them makes either only right turns or only left turns. A simple polygon P is said to be *weakly convex visible* from a line segment inside P, if every point in the polygon is convex visible from some point of the line segment. In this paper we propose an algorithm for recognizing weakly convex visible polygons. Our algorithm computes a line segment inside the given polygon from which the polygon is weakly convex visible. For an n-sided polygon, the algorithm runs in $O(n^2 \log n)$ time. © 1998 Elsevier Science B.V.

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1. Introduction

In recent years, visibility problems have been studied extensively in computational geometry. O'Rourke's monograph [13] contains a detailed exposition of earlier results on visibility problems. Shermer has written a survey of more recent results [14]. Two points inside a simple polygon are said to be *visible* if the line segment joining them is not crossed by any edge of the polygon. Lee and Preparata [12] proposed a linear time algorithm for determining the set of all points inside a simple polygon, where each point in the set is visible from every point in the polygon. Such a set is called the *kernel* of the polygon. El Gindy and Avis [4], Lee [11] and Joe and Simpson [9] designed linear time algorithms for determining the region of a given simple polygon visible from a given point inside the polygon. In [1], Avis and Toussaint introduced the notion of *weak visibility* from a line segment; a simple polygon is *weakly visible* from a line segment inside the polygon, if every point inside the polygon is visible from some point inside the line segment. Ke [10], Ghosh et al. [5] and Das et al. [3]

* Corresponding author. E-mail: spp@cse.iitkgp.ernet.in.

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proposed algorithms for determining an internal line segment (if any), such that the polygon is weakly visible from the line segment.

Consider the generalization of visibility where two points inside a simple polygon are said to be *convex visible* if the Euclidean shortest path between them makes either only left turns or only right turns. The *palm kernel* is the set of all points of a given simple polygon such that each point in the set is convex visible from every point in the polygon. The palm kernel can be computed in $O(E)$ time, where E is the size of the visibility graph of the given polygon (see [6]). A polygon is called *weakly convex visible* from a line segment lying inside the polygon, if every point in it is convex visible from some point inside the line segment. In this paper, we design an $O(n^2 \log n)$-time algorithm for determining whether a given n-gon is weakly convex visible from an internal line segment. Our algorithm identifies a line segment (if one exists), such that the given polygon is weakly convex visible from the line segment. Such a segment is called a *convex visibility segment.* In order to derive this result we characterize weakly convex visible polygons and establish several relevant and interesting properties of weak convex visibility. These results were published earlier in a preliminary version of this paper (see [2]). In [2], we had also proposed an algorithm for computing the *maximum hidden set* of the *convex visibility graph* for a very restricted class of polygons.

In Section 2 we state a few definitions, introduce some notation, and develop certain preliminary results required throughout the paper. In Section 3, we design an $O(n^2 \log n)$ time algorithm for computing a convex visibility segment (if one exists) inside a given n-gon. In Section 4 we conclude with a few remarks.

2. Definitions, notation and preliminaries

We assume that the simple polygon P is given as a counterclockwise sequence of *vertices* $v_1, v_2, \ldots, v_n$ with their respective x- and y- coordinates. We assume that no three vertices of P are collinear. The line segments $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$ are called edges of P. The symbol P is used to denote the interior and the boundary of the polygon P. If p and q are two points on $\mathrm{bd}(P)$ then the part of $\mathrm{bd}(P)$ traversed from p to q in counterclockwise fashion is denoted by $\mathrm{bd}(p, q)$.

Two points are said to be *convex visible* if the Euclidean shortest path joining them has only right turns or only left turns. A point p is said to be *weakly convex visible* from an edge or an internal segment st, if there is a point z in the *interior* of st such that p and z are convex visible. If every point of P is weakly convex visible from st then P is said to be *weakly convex visible* from st. If a polygon is weakly convex visible from an internal segment st, we say that st is a *convex visibility segment.* A region of P is called the *weakly convex visible from st* if each point in that region is weakly convex visible from st. We use $\mathrm{SP}(u, v)$ to denote the Euclidean shortest path inside P from a point u to another point v. For any vertex u of P the *shortest path tree* of P *rooted at* u, denoted as $\mathrm{SPT}(u)$, is the union of the shortest paths from u to all the vertices of P.

Given any three points $p_i = (x_i, y_i)$, $p_j = (x_j, y_j)$ and $p_k = (x_k, y_k)$, let

$$S = x_k(y_i - y_j) + y_k(x_j - x_i) + y_jx_i - y_ix_j.$$

If $S < 0$, then $p_ip_jp_k$ is a right turn. If $S > 0$, then $p_ip_jp_k$ is a left turn. If S is zero, then the three points are collinear.

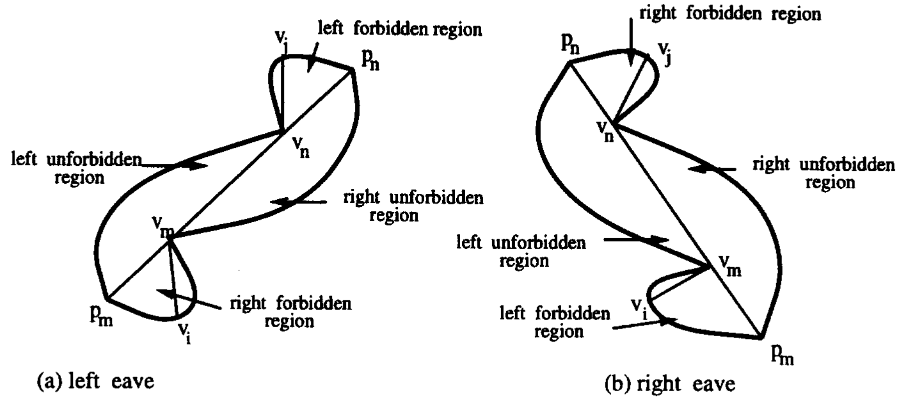

Fig. 1. Four regions of the eave $v_m v_n$.

The shortest path SP(v_i, v_j) is represented as a sequence of vertices $v_i, \ldots, v_j$. If SP(v_i, v_j) has left as well as right turns, then it makes a left (right) turn followed by a right (left) turn on some edge of SP(v_i, v_j). Such an edge is called an *eave* of the shortest path. An *eave* $v_k v_l$ of SP(v_i, v_j) is called *left* (*right*) *eave*, if v_k precedes v_l in the sequence of vertices of SP(v_i, v_j), and the shortest path makes a *right* (*left*) turn at v_k and a *left* (*right*) turn at v_l.

Consider two paths SP(v_i, p) and SP(v_i, q), where p and q are points inside P. These paths share a common part starting at v_i and then separate out at a vertex v_j, never to meet again. Let $v_j s_1$ ($v_j s_2$) be the first edge of SP(v_j, p) (SP(v_j, q)). Let e and f be eaves on the disjoint parts of SP(v_i, p) and SP(v_i, q), respectively. If $v_j s_2 s_1$ is a left (right) turn, then we say that e *lies to the left* (*lies to the right*) of f in SPT(v_i). Let E be a set of eaves of SPT(v_i) such that no two eaves of E lie on the same root to leaf path of SPT(v_i). Then, the eave of E that lies to the left (right) of every other eave is called the *leftmost eave* (*rightmost eave*) of E.

Let $v_m v_n$ be a left eave of SP(v_i, v_j). The *extension* $p_m p_n$ of the eave $v_m v_n$ is the *chord* of P that intersects the boundary of the polygon at p_m and p_n (Fig. 1(a)). (A *chord* is a maximal length segment that lies entirely within P.) We define the *forbidden region* for the left eave $v_m v_n$; the *right forbidden region* (*left forbidden region*) is the region bounded by bd(p_m, v_m) (bd(p_n, v_n)) and $v_m p_m$ ($v_n p_n$). The remaining region of P is called the *unforbidden region*; the *right unforbidden region* (*left unforbidden region*) is the region bounded by bd(v_m, p_n) (bd(v_n, p_m)) and $p_n v_m$ ($p_m v_n$). Similar and symmetric regions can be defined for a right eave (see Fig. 1(b)).

The following properties are immediate from the definition of eaves.

Proposition 2.1. *If the shortest path between two points has two left* (*right*) *eaves then there must be a right* (*left*) *eave in the path between the two left* (*right*) *eaves.*

Proposition 2.2. *If the first turn in the shortest path between two points is a left* (*right*) *turn, then the first eave in the shortest path must be a right* (*left*) *eave.*

We characterize convex visibility segments inside a simple polygon in the following theorem.

Theorem 2.3. *An internal segment st of a simple polygon P is a convex visibility segment if and only if st intersects the unforbidden region of every eave of P.*

Proof. We establish the contrapositive of the if-part as follows. Suppose there exists a point, say p, which is not weakly convex visible from an internal segment st. Let q be any point on st. Since p is not convex visible from q, the shortest path $\mathrm{SP}(p,q)$ is not convex and must therefore have an eave, say e. Clearly, q lies inside one forbidden region of e and p lies in the other forbidden region of e. The segment st lying entirely in a forbidden region of eave e, can not intersect the unforbidden region of e.

We prove the contrapositive of the only-if-part as follows. We assume that an internal segment st does not intersect the unforbidden region of some eave, say e. This implies that st lies in one of the two disjoint forbidden regions of e. Let p be any point in the other forbidden region of e. Then, $\mathrm{SP}(p,q)$, where $q \in st$, is not convex. Thus the point p is not weakly convex visible from st. □

We say that an eave *separates* an internal line segment from a point if the line segment lies entirely in the interior of one forbidden region of the eave and the point lies in the other forbidden region of the eave. Note that if an eave separates a line segment from a point then the point is not weakly convex visible from the line segment. Moreover, the line segment and the extension of the separating eave, do not meet. We have the following result, whose proof requires arguments similar to those used in the proof of the if-part of Theorem 2.3.

Corollary 2.4. *If a point p inside a simple polygon P is not weakly convex visible from an internal segment st, then there is an eave e which separates p from st. Moreover, all points lying in the forbidden region of e containing p (including at least one vertex of P), are separated from st by e.*

Lemma 2.5. *If a point p in the forbidden region of an eave e is not weakly convex visible from the extension of the eave e, then there is a point q in P such that $\mathrm{SP}(p,q)$ has two eaves, whose extensions do not meet inside P.*

Proof. Using Corollary 2.4 we know that there is an eave f separating e from p. Consider any point q in that forbidden region of e which does not contain p. Then, $\mathrm{SP}(p,q)$ has two eaves, e and f, whose extensions do not meet inside P. □

We conclude this section with a simple linear time algorithm for computing the portion of P, weakly convex visible from an internal line segment st. From Corollary 2.4, we know that any point p of P, which is not weakly convex visible from st, is separated from st by an eave. Such an eave must be common to both, $\mathrm{SP}(s,p)$ and $\mathrm{SP}(t,p)$. If there are several such common eaves, then choose the one that is closest to s on $\mathrm{SP}(s,p)$. Every point in the forbidden region of this eave including p, is not weakly convex visible from st. So, we delete this forbidden region from P. Note that it is sufficient to delete all such forbidden regions from P in order to compute the portion of P which is weakly convex visible from st. Such forbidden regions are determined by the nearest common eaves between $\mathrm{SPT}(s)$ and $\mathrm{SPT}(t)$, on the paths of $\mathrm{SPT}(s)$. Once such an eave is identified, its forbidden region can be found and deleted by computing the extension of the eave using the *shortest path map* $\mathrm{SPM}(s)$ (see [8]). The shortest path map $\mathrm{SPM}(s)$ can be computed in linear time and contains the extension to $\mathrm{bd}(P)$ of each edge of $\mathrm{SPT}(s)$. The shortest path trees can also be computed in linear time [7], and the common eaves can be determined by simple breadth first searches on the shortest path trees. We have the following result.

Theorem 2.6. *Given a simple polygon P of n vertices and a line segment st inside P, it is possible to compute in linear time, the portion of P which is weakly convex visible from st.*

Corollary 2.7. *Given a simple polygon P with n vertices and a line segment st inside P, the set of all eaves separating points in P from st, can be determined in linear time.*

3. Algorithm for recognizing weakly convex visible polygons

In this section, we present an $O(n^2 \log n)$ time algorithm for computing a convex visibility segment (if one exists) inside a simple polygon P of n vertices. To begin with, we compute shortest path maps rooted at every vertex of P (see [8]). The map data structure not only stores the respective shortest path tree, but also the extensions to $\mathrm{bd}(P)$ of the edges of the shortest path tree. Since eaves are edges of shortest path trees, their extensions are easily determined from the maps.

We distinguish between the following mutually exclusive and exhaustive cases.

Case 1. Every shortest path in P has at most one eave.

Case 2. Every shortest path in P has at most two eaves, and at least one shortest path in P has two eaves. Moreover, in every shortest path with two eaves, the extensions of the two eaves intersect (inside the polygon).

Case 3. At least one shortest path in P has two eaves whose extensions do not intersect.

We show that there is always a convex visibility segment in Cases 1 and 2. In Case 3, there may or may not be a convex visibility segment. In Cases 1 and 2, the proofs of existence of convex visibility segments are of constructive nature. We show that such a segment can be located inside any polygon belonging to these cases. This search however, is not straightforward and requires consideration of several systematically defined cases. In Case 3, the search starts between the non-intersecting extensions of the two eaves on a single shortest path. The convex visibility segment that our algorithm computes is always a chord of P.

We consider each of these cases separately.

Case 1

Consider $\mathrm{SPT}(v_i)$, where v_i is an arbitrary vertex. We have the following subcases:

Case 1.1. All eaves of $\mathrm{SPT}(v_i)$ are left eaves.

Case 1.2. All eaves of $\mathrm{SPT}(v_i)$ are right eaves.

Case 1.3. $\mathrm{SPT}(v_i)$ has left as well as right eaves.

Since Cases 1.1 and 1.2 are symmetrical, it is sufficient to consider only Case 1.1 and Case 1.3.

Consider Case 1.1. $\mathrm{SPT}(v_i)$ has only left eaves. We need a few definitions. We call the vertex v_i the degenerate segment ll_0. Clearly, if $\mathrm{SPT}(v_i)$ has no eave, then P is weakly convex visible from ll_0. Otherwise, let E_0 denote the set of eaves of $\mathrm{SPT}(v_i)$. In Fig. 2, $E_0 = \{e_1, e_2, e_3\}$. Let ll_1 denote the leftmost left eave of $\mathrm{SPT}(v_i)$ in E_0. We define E_i as follows. Let

$$E_i = \{e \mid \text{eave } e \text{ separates a point } p \text{ from the extension of } ll_i \text{ where } p \text{ lies in the right unforbidden region of } ll_i\}, \quad 1 \leqslant i.$$

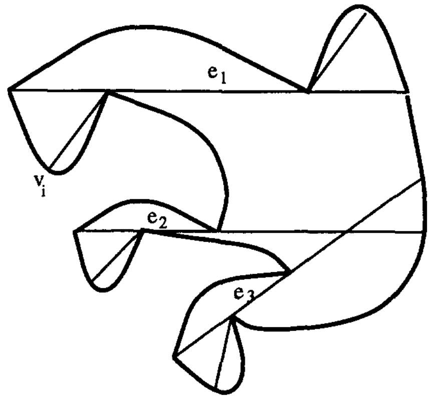

Fig. 2. E_2 is empty. $ll_2 = e_3$ is the convex visibility segment.

Assuming E_i is not empty, let ll_{i+1} be the leftmost left eave in E_i, $1 \leqslant i$. In Fig. 2, $ll_1 = e_1$, $E_1 = \{e_2, e_3\}$, $ll_2 = e_3$ and E_2 is empty. Following the definition of E_i, note that ll_{i+1} separates the extension of ll_i from some point in the right unforbidden region of ll_i. Therefore, the extension of ll_{i+1} does not meet that of ll_i, $0 \leqslant i$. Also note that $E_i \subset E_{i-1}$, $0 \leqslant i$. Consider the finite sequence of sets $E_0, E_1, E_2, \ldots, E_r$, where $E_r = \phi$ and $E_{r-1} \neq \phi$. This sequence is finite because P has a finite number of edges. We claim that ll_r is a convex visibility segment of P.

Lemma 3.1. *Consider the finite sequence of sets $E_0, E_1, E_2, \ldots, E_r$, where $E_r = \phi$ and $E_{r-1} \neq \phi$. Then ll_r, the leftmost left eave of E_{r-1}, is a convex visibility segment of P.*

Proof. Since P has at most one eave in every shortest path, by Lemma 2.5, every point in the left and right forbidden regions of ll_r is weakly convex visible from its extension. Since E_r is empty, the entire right unforbidden region of ll_r is weakly convex visible from its extension.

It remains to show that every point in the left unforbidden region of ll_r is *weakly convex visible* from its extension. Suppose some point p in the left unforbidden region of ll_r is not weakly convex visible from its extension. By Corollary 2.4, there must be an eave e separating p from the extension of ll_r. Since the extensions of ll_{r-1} and ll_r do not meet and both ll_r and p lie in the right unforbidden region of ll_{r-1}, e must also separate the extension of ll_{r-1} from p. Therefore, by the definition of E_{r-1}, $e \in E_{r-1}$. However, p is in the left unforbidden region of ll_r. So, e is the leftmost left eave in E_{r-1}, a contradiction to the fact that ll_r ($\neq e$) is the leftmost left eave in E_{r-1}. □

The set E_0 can be computed in linear time using the shortest path tree algorithm of [7]. Computation of each set E_i, $1 \leqslant i \leqslant r$, can be done in linear time (see Corollary 2.7). The computation of ll_{i+1} from E_i in linear time is straightforward. So, we can find the convex visibility segment ll_r in $O(n^2)$ time. This completes the discussion on Case 1.1.

Consider Case 1.3. We first show that each right eave of SPT(v_i) must lie to the left of each left eave of SPT(v_i). The following lemma is applicable also for polygons in Case 2.

Lemma 3.2. *If* SP(v_i, v_j) *has only a left eave and no right eave and* SP(v_i, v_k) *has only a right eave and no left eave, then $v_j \in$* bd(v_i, v_k)*. In other words, the right eave in* SP(v_i, v_k) *lies to the left of the left eave in* SP(v_i, v_j).

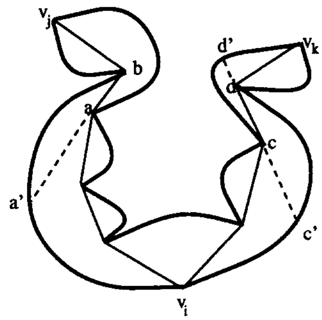

Fig. 3. SP(v_k, v_j) has two eaves.

Proof. Let ab (cd) be the left (right) eave in SP(v_i, v_j) (SP(v_i, v_k)). For the sake of contradiction, suppose $v_k \in$ bd(v_i, v_j). If v_k lies in bd(a, v_j), then SP(v_i, v_k) coincides with SP(v_i, v_j) at least till the vertex a and thus takes a right turn at a. So, by Proposition 2.2, SP(v_i, v_k) must have a left eave. This is a contradiction because SP(v_i, v_k) has only a single right eave and no left eave.

Now assume that v_k lies in bd(v_i, a) (see Fig. 3). Since there is only one eave in SP(v_i, v_k) (SP(v_i, v_j)), SP(v_i, d) (SP(v_i, b)) must make only left (right) turns. So, the backward extension of the eave cd (ab) will intersect bd(P) at a point c' (a') that lies in bd(v_i, d) (bd(b, v_i)). So, $c'd$ ($a'b$) lies in the region bounded by bd(v_i, d) (bd(b, v_i)) and the left (right) turning SP(v_i, d) (SP(v_i, b)). Clearly, $c'd$ and $a'b$ lie in disjoint parts of P. Thus, the extensions of the eaves ab and cd do not meet. So, SP(v_j, v_k) has two eaves ab and cd, whose extensions do not meet, a contradiction. □

The above lemma shows that in SPT(v_i), each right eave lies to the left of each left eave. Exactly as in Case 1.1, consider the finite sequence of sets $E_0, E_1, E_2, \ldots, E_r$, where $E_r = \phi$ and $E_{r-1} \neq \phi$. Again, as in Case 1.1, we claim that ll_r, the leftmost left eave of E_{r-1}, is a convex visibility segment of P.

Note that E_0 has right and left eaves, whereas each E_i, $1 \leqslant i$, can have only left eaves. Suppose E_1 is not empty. Since the extension of ll_r does not meet that of ll_1, the extension of ll_r lies entirely in the right unforbidden region of ll_1. So, all the right eaves of SPT(v_i) lie in a forbidden region ll_r. By Lemma 2.5, every point in the two forbidden regions ll_r is convex visible from the extension of ll_r. Since E_r is empty, the entire right unforbidden region of ll_r is also weakly convex visible from its extension. In order to prove that every point in the left unforbidden region of ll_r is weakly convex visible from the extension of ll_r, we can use arguments similar to those in the proof of Lemma 3.1.

Now assume that E_1 is empty (see Fig. 4). So, by the definition of E_1, it is clear that the right unforbidden region of ll_1 is weakly convex visible from the extension of ll_1. The forbidden regions of ll_1 are also weakly convex visible from the extension of ll_1 (see Lemma 2.5). We now show that the left unforbidden region of ll_1 is also weakly convex visible from the extension of ll_1. If it were not so, then there must be an eave e that separates the extension of ll_1 from p (see Corollary 2.4). This eave e must also separate v_i and the entire left forbidden region of ll_1 from p. Since e separates p from v_i, SP(v_i, p) must have eave e. Since each shortest path has at most one eave and p lies in the left unforbidden region of the leftmost left eave ll_1 of SPT(v_i), e must be a right eave. Now consider any point q in the left forbidden region of ll_1. The eave e also separates p from the entire left forbidden region of ll_1. So, SP(q, p) takes a right turn at a vertex of ll_1 and then passes through the right eave e.

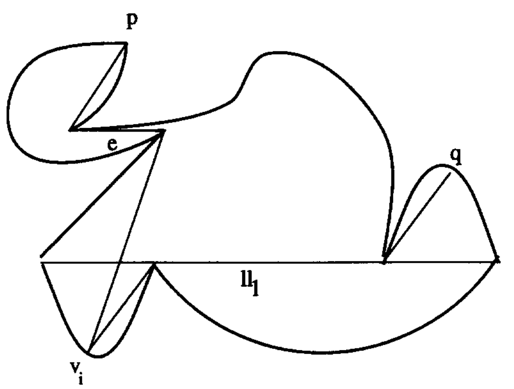

Fig. 4.

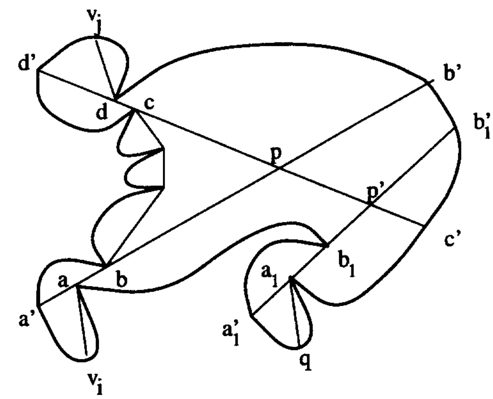

Fig. 5. $\mathrm{bd}(a_1', a_1)$ is not weakly convex visible from $a'b'$.

By Proposition 2.2, $\mathrm{SP}(q, p)$ must have a left eave too, in addition to the right eave e, a contradiction to the assumption that every shortest path has at most one eave.

This concludes the discussion on Case 1.3.

Case 2

Let v_i and v_j be two vertices such that $\mathrm{SP}(v_i, v_j)$ has two eaves ab and cd. We assume without loss of generality that ab (cd) is a left (right) eave and ab precedes cd as we traverse $\mathrm{SP}(v_i, v_j)$ from v_i to v_j. Let p be the point of intersection of the extensions $a'b'$ and $c'd'$, of the eaves ab and cd, respectively (see Fig. 5). The region bounded by $\mathrm{bd}(c', b')$ and the segments $b'p$ and pc' is the common unforbidden region of the two eaves ab and cd. We say that the region bounded by $\mathrm{bd}(a, c')$, $c'p$ and pa is the *critical unforbidden* region of ab. We have the following lemma.

Lemma 3.3. *Let* $\mathrm{SP}(v_i, v_j)$ *be a shortest path with two eaves ab and cd. Then, every point in the forbidden regions of the ab, and the left unforbidden region of ab, is weakly convex visible from the extension* $a'b'$ *of ab.*

Proof. Consider Fig. 5. The extensions $a'b'$ and $c'd'$ of ab and cd, intersect at a point p within the polygon P. By Lemma 2.5 we know that every point in the forbidden regions of the eave ab must be weakly convex visible from $a'b'$. Now consider the left unforbidden region of ab. Suppose a point q in this region is not weakly convex visible from $a'b'$. So, by Corollary 2.4, we know that there is an eave e that separates q from $a'b'$, and the extension of e does not meet $a'b'$. So, the extension of e can not intersect $c'd'$. Considering the shortest path $\mathrm{SP}(v_j, q)$, there are two eaves viz., cd and e and their extensions do not meet, a contradiction. □

It is possible that there exist points in the critical unforbidden region of ab which are not weakly convex visible from $a'b'$. We have the following lemma.

Lemma 3.4. *If a point q in the critical unforbidden region of the eave ab is not weakly convex visible from the extension* $a'b'$, *then* $\mathrm{SP}(v_i, q)$ *has a unique left eave that separates q from* $a'b'$.

Proof. Consider Fig. 5. Since q is not weakly convex visible from $a'b'$, we know from Corollary 2.4 that there is an eave e separating $a'b'$ from q, and the extension of e does not meet $a'b'$. So, e also separates v_i and v_j from q. Note that $\mathrm{SP}(v_j, q)$ has at least two eaves, viz., the right eave cd and e. If e were a right eave then $\mathrm{SP}(v_j, q)$ would have a left eave too (see Proposition 2.1), a contradiction to the assumption that each shortest path has at most two eaves. So, e must be a left eave. Since e also separates v_i from q, $\mathrm{SP}(v_i, q)$ must have left eave e. Note that there can be at most one left eave or at most one right eave in any shortest path in polygons comprising Case 2. This implies that the eave e is unique. □

Using Lemma 3.4, we define the set $F = \{e \mid \text{eave } e \text{ is the unique eave separating a point } q$ of the critical unforbidden region of ab from $a'b'\}$. The set F can be computed in linear time (see Corollary 2.7). Let a_1b_1 be the leftmost left eave in F. Note that the extension of a_1b_1 must meet $c'd'$, at say p' (see Fig. 5). Otherwise, $\mathrm{SP}(v_j, q)$ would have two eaves cd and a_1b_1, whose extensions do not intersect, a contradiction. The eave a_1b_1 inherits certain properties of the left eave ab. Observe that Lemma 3.3 applies to a_1b_1 as it applies to ab, with any vertex v of P in $\mathrm{bd}(a'_1, a_1)$ playing the role of v_i. More precisely, the forbidden regions of a_1b_1 and the left unforbidden region of a_1b_1 are weakly convex visible from $a'_1b'_1$. Further, we can show that the entire critical unforbidden region of a_1b_1, i.e., the region bounded by a_1p', $p'c'$ and $\mathrm{bd}(a_1, c')$ is also weakly convex visible from $a'_1b'_1$. This can be done as in the proof of Lemma 3.1, by using the property that a_1b_1 is the leftmost eave in F.

So far we have established that every point outside the common unforbidden region of a_1b_1 and cd is weakly convex visible from the extension $a'_1b'_1$. We call any vertex v in $\mathrm{bd}(a'_1, a_1)$ as v_i and rename a_1, b_1, a'_1, b'_1 and p', as a, b, a', b' and p, respectively.

We now concentrate on the common unforbidden region of ab and cd, where it is possible that some point q is not weakly convex visible from $a'b'$ (see Fig. 6). So, $\mathrm{SP}(a, q)$ must have an eave e. Let $S = \{s \mid s \text{ is a shortest path } \mathrm{SP}(a, q)$, where q is a point in the common unforbidden region of ab and cd, and q is not weakly convex visible from $a'b'\}$. Let $E = \{e \mid e \text{ is an eave of } s \in S\}$.

Two cases arise.

Case 2.1. All shortest paths in S have only one eave.

Case 2.2. A path in S has two eaves.

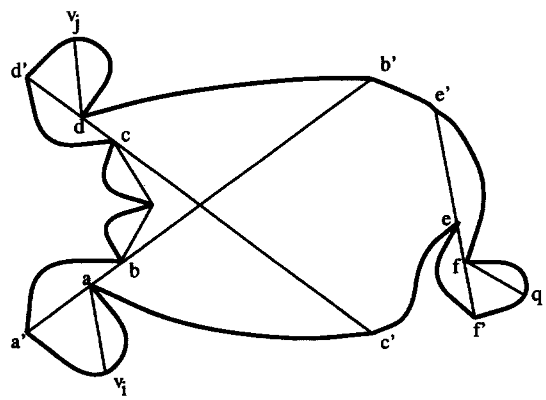

Fig. 6. Common unforbidden region contains only left eaves.

Case 2.1

We have the following subcases.

Case 2.1.1. The eave in each shortest path in S is a left eave.

Case 2.1.2. The eave in each shortest path in S is a right eave.

Case 2.1.3. The eave in a shortest path in S is either a left eave or a right eave.

Since Cases 2.1.1 and 2.1.2 are similar and symmetric, we consider only Cases 2.1.1 and 2.1.3.

Consider Case 2.1.1. We claim that the extension of the leftmost left eave ef in E is the convex visibility segment (see Fig. 6). By Lemma 2.5, every point in the left and right forbidden regions of the eave ef is weakly convex visible from its extension.

Consider the left unforbidden region of ef. By the definition of E, there is a point q in the common unforbidden region of the eaves ab and cd, such that $\mathrm{SP}(a, q)$ has an eave ef, separating q from the extension of ab. So, the extension of ef does not meet $a'b'$. Therefore, the left unforbidden region of ef lies entirely in the common unforbidden region of the eaves ab and cd (Fig. 6). Suppose some point r in the left unforbidden region of ef is not weakly convex visible from the extension of ef. By Corollary 2.4, there must be an eave l separating the extension $e'f'$ of ef from r. Since $e'f'$ does not meet $a'b'$, l also separates a point r in the left unforbidden region of ef from $a'b'$. Therefore, l ($\neq ef$) belongs to E, and is to the left of the leftmost eave ef in E, a contradiction.

Now consider the right unforbidden region of ef. Suppose a point r in this region is not weakly convex visible from the extension $e'f'$ of ef. Then, by Corollary 2.4 we know that there is an eave gh that separates r from $e'f'$. Note that r is not weakly convex visible also from $a'b'$. So, gh belongs to E. The eave gh also separates r from $a'b'$, v_i and v_j. So, $\mathrm{SP}(v_j, r)$ must have eave gh. Now $\mathrm{SP}(v_j, r)$ takes a left turn at d. We know that $a'b'$ and $e'f'$ do not meet. Also v_j lies in the right forbidden region of ef and r lies in the right unforbidden region of ef. So, $\mathrm{SP}(v_j, r)$ takes a right turn at e. Therefore, there must be a right eave on $\mathrm{SP}(v_j, r)$ between vertices d and e, whose extension does not enter the right forbidden region of ef. The other eave gh of $\mathrm{SP}(v_j, r)$ lies entirely inside the right forbidden region of ef because it does not meet $e'f'$. So, there are two eaves in $\mathrm{SP}(v_j, r)$ whose extensions do not meet, a contradiction.

Now consider Case 2.1.3. Let ef (gh) be a left (right) eave in E (see Fig. 7). Then, there are points q and r in the common unforbidden region of ab and cd, such that ef (gh) separates q (r) from the extension of ab. By Corollary 2.4, we know that there are vertices v_j and v_k, such that $\mathrm{SP}(a, v_j)$ has only one left eave ef and no right eave and, $\mathrm{SP}(a, v_k)$ has only one right eave gh and no left eave. Lemma 3.2 implies that gh is to the left of ef. Let ef be the leftmost left eave in E; all right eaves in E are to the left of ef. The proof that the right unforbidden region of ef is weakly convex visible from the extension $e'f'$ of ef is identical to the proof of the similar fact in Case 2.1.1. However, $e'f'$ is not a convex visibility segment if some point q in the left unforbidden region of ef is not weakly convex visible from $e'f'$ (Fig. 7). In such a case, we know by Corollary 2.4, that an eave say gh, separates q from $e'f'$. Clearly, gh is to the left of ef. Since $a'b'$ and $e'f'$ do not meet, gh also separates q from $a'b'$. So, gh belongs to E. Since ef is the leftmost left eave of E, and gh lies to the left of ef, gh must be a right eave. We claim that the extension of the rightmost right eave in E is a convex visibility segment.

Let gh be the rightmost right eave in E. By Lemma 2.5, the two forbidden regions of gh are weakly convex visible from the extension of gh. The proof that the left unforbidden region of gh is weakly convex visible from the extension of gh is similar and symmetric to the proof that the right unforbidden region of the leftmost left eave in E in Case 2.1.1 is weakly convex visible from its extension.

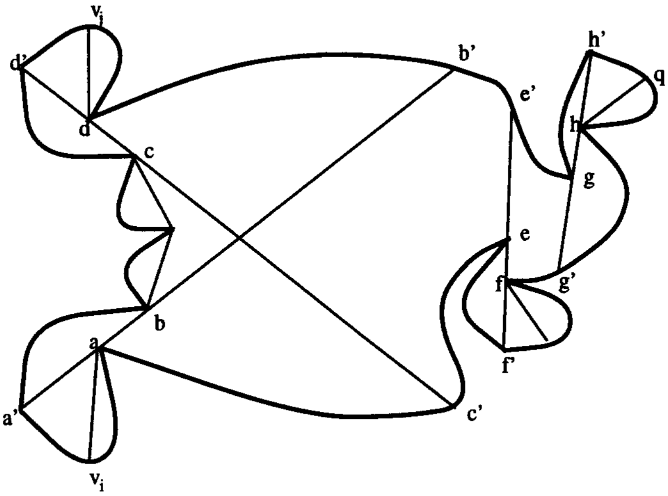

Fig. 7. The left unforbidden region of ef is not convex visible from $e'f'$.

Now consider the right unforbidden region of gh. Since gh separates a point q in the left unforbidden region of ef from $e'f'$, the extension of gh does not meet $e'f'$. So, the right unforbidden region of gh clearly lies in the left unforbidden region of ef. If there is a point p in the right unforbidden region of gh that is not weakly convex visible from its extension, then it will also not be weakly convex visible from $a'b'$ causing an eave in E to separate it from $a'b'$. This is a contradiction because such an eave must lie to the left of the leftmost left eave ef in E and to the right of the rightmost right eave gh in E.

Consider Case 2.2. In this case, there is a path $\mathrm{SP}(a,q) \in S$ that has exactly two eaves. The path $\mathrm{SP}(v_i,q)$ passes through the vertex a and takes a right turn at a. By Proposition 2.2, the first eave in $\mathrm{SP}(v_i,q)$ must be a left eave. The first left eave in $\mathrm{SP}(v_i,v_j)$ is ab. So, $\mathrm{SP}(v_i,a)$ makes only right turns. Therefore, the first left eave of $\mathrm{SP}(v_i,q)$ must also be the first eave of $\mathrm{SP}(a,q)$. So, s has a left eave followed by a right eave.

Let $S' = \{s \mid s \in S$ has exactly two eaves$\}$. Let ef (gh) be the left (right) eave in the rightmost path $\mathrm{SP}(a,q)$ of S'. Note that there are no right eaves of $\mathrm{SPT}(a)$ to the right of this path. In Case 2, we know that the extensions of ef and gh must intersect (see Fig. 8). Now by Lemma 3.3, every point in the two forbidden regions of ef, and the left unforbidden region of ef must be weakly convex visible from the extension $e'f'$ of the ef.

If the right unforbidden region of ef is weakly convex visible from $e'f'$, then $e'f'$ is a convex visibility segment. Suppose some point q in the right unforbidden region of ef is not weakly convex visible from $e'f'$. By Corollary 2.4, there exists an eave l that separates q from $e'f'$. This eave must be a left eave because there are no right eaves of $\mathrm{SPT}(a)$ to the right of the path in S' having eaves ef and gh. We define a set $F = \{l \mid l$ is an eave that separates q from the extension $e'f'$ of ef, where q belongs to the right unforbidden region of $ef\ \}$. We claim that the extension of the leftmost eave in F is a convex visibility segment (in Fig. 8, rt is the leftmost left eave in the right unforbidden region of ef). Proof is similar to that of Case 2.1.1.

This concludes the discussion on Case 2.2.

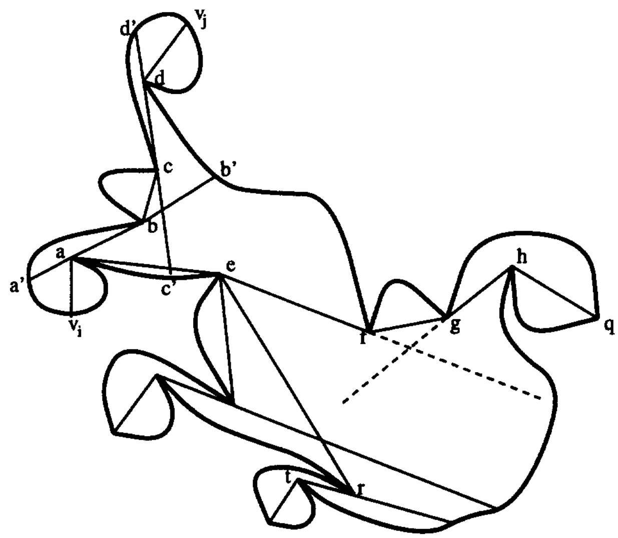

Fig. 8. ef is the left eave on the rightmost path in S'.

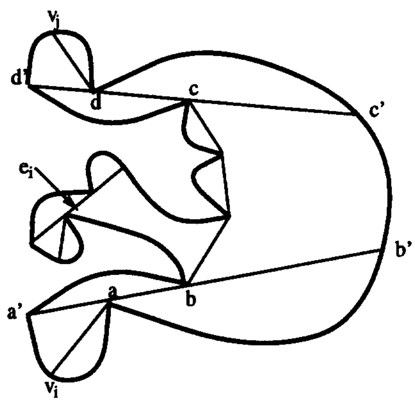

Fig. 9. H_{i-1} is in the unforbidden region of e_i.

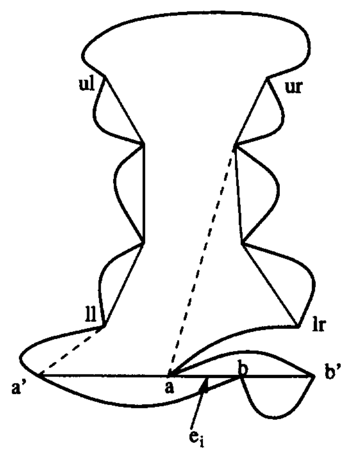

Fig. 10. H_{i-1} lies in the forbidden region of e_i.

Case 3

We start with a shortest path SP(v_i, v_j) having two eaves ab and cd whose extensions do not meet (Fig. 9). Due to Theorem 2.3, we know that a convex visibility segment must cross the extensions of ab and cd (segments bb' and cc' in Fig. 9). This gives us a suitable starting point for searching a convex visibility segment. Before proceeding further, we state the following definitions.

We define an *hourglass* by its four *corners*. The corners are points say, ll, lr, ur and ul, appearing in counterclockwise order on bd(P) (see Fig. 10). We say that SP(ll, ul) and SP(lr, ur) are the two *sides* of the hourglass. We call bd(ll, lr) and bd(ur, ul) the two *bases* of the hourglass. The hourglass is also used to represent the region of P enclosed by its two sides and two bases. Any chord of P that connects bd(ll, lr) and bd(ur, ul) is said to *span* the hourglass.

Consider the hourglass H_0 defined by corners b, b', c and c' in Fig. 9. Every chord spanning H_0 may not stab the unforbidden regions of the other eaves of P, and therefore, may not be a convex visibility segment. So, we need to *process* all the eaves of P. Let E denote the set of eaves of P excluding ab and cd. We *process* eaves of E iteratively, in arbitrary order. Let e_i be the eave processed in the ith iteration. We modify the hourglass H_{i-1} to obtain H_i, $i \geqslant 1$, depending upon how e_i interacts with H_{i-1}. We use E_i to denote the subset of E containing those eaves which have been processed already, and whose unforbidden regions are stabbed by both the cross tangents of H_i. A *cross tangent* is a chord of P that is a tangent to both sides of the hourglass. The invariant of the iteration is that both the cross tangents of H_i stab the unforbidden regions of all the eaves of E_i. Although $e_i \in E_i \setminus E_{i-1}$, the cardinality of E_i may be less than that of E_{i-1}; during the ith iteration, several eaves may be taken out of E_{i-1} and sent back for reprocessing. However, we can show that each eave may be reprocessed at most once. So, the algorithm terminates.

As we shall see later in this section, the processing of e_i and the updating of H_{i-1} to H_i, involves a possible modification of one of the bases of H_{i-1}. Although the hourglass is updated in each iteration, we do not construct its sides until all eaves are processed. If the two sides of the final hourglass H share a vertex, we say that H has *collapsed* and the polygon does not have a convex visibility segment. Otherwise, any cross tangent of H is a convex visibility segment because both its cross tangents stab the unforbidden regions of all eaves in E (see Theorem 2.3).

The processing of eave e_i is indeed very simple if its extension connects both the bases of H_{i-1}; setting $H_i = H_{i-1}$ preserves the invariant. Even if the extension of e_i runs across H_{i-1}, intersecting its two sides, simply setting $H_i = H_{i-1}$ preserves the invariant. The only remaining possibilities are that the extension of e_i may

(i) lie entirely within the region bounded by only one side of H_{i-1} and bd(P) (Fig. 9),
(ii) intersect only one base and no side of H_{i-1} (Fig. 10),
(iii) intersect one side and one base of H_{i-1} (Fig. 11).

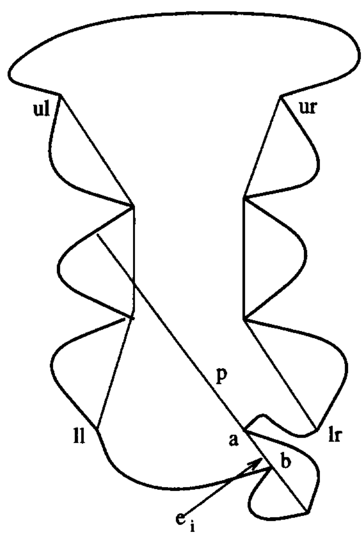

Fig. 11. One base is entirely in the forbidden region of e_i.

In case (i), the entire hourglass may lie within a forbidden region of e_i (Fig. 9); in this case we clearly can not have a convex visibility segment because such a segment must not only span the hourglass, but also stab the unforbidden region of e_i, an impossibility. Otherwise, the entire hourglass lies in the unforbidden region of e_i. So, the unforbidden region of e_i is stabbed by every chord spanning H_{i-1}, and setting $H_i = H_{i-1}$ preserves the invariant.

In case (ii), if the forbidden region of e_i encloses the other base and the two sides of H_{i-1} (see Fig. 10), we update H_{i-1} to get H_i by shrinking the base intersected by the extension of e_i. The new base is defined by a part of the extension of e_i and the other base inside the forbidden region of e_i remains unchanged. In Fig. 10, the base $\mathrm{bd}(ll, lr)$ is shrunk to $\mathrm{bd}(a, a')$. Clearly, the unforbidden region of e_i is stabbed by both the cross tangents of H_i. Applying the invariant we know that the two cross tangents of H_{i-1} touch the unforbidden regions of all eaves in E_{i-1}. Those eaves of E_{i-1} whose extensions touch both the bases of H_{i-1} but do not touch the newly modified base of H_i must be sent for reprocessing. So, we obtain $E_i = E_{i-1} \setminus \{f \mid f \in E_{i-1}$ and f does not touch the new base of $H_i\} \cup \{e_i\}$. Thus, the invariant holds at the end of the ith iteration. The other possibility in case (ii) is that the unforbidden region of e_i encloses the other base and the two sides of H_{i-1}. Clearly, the assignments $H_i = H_{i-1}$ and $E_i = E_{i-1} \cup \{e_i\}$, preserve the invariant.

In case (iii), one base A of H_{i-1} may lie entirely in the forbidden region of e_i (see Fig. 11). The other base B of H_{i-1} is partly in the forbidden region of e_i. We need to modify H_{i-1} as follows. We traverse and discard the portion of the B lying in the forbidden region of e_i, thereby shrinking the base B to create H_i with this modified base and the base A. As in case (ii), we collect eaves of E_{i-1} whose extensions do not meet the new base of H_i and, send them for reprocessing. The remaining eaves along with e_i constitute the set E_i, and the invariant holds. The other possibility in case (iii) is that one entire base A of H_{i-1} is in the unforbidden region of e_i. Then, clearly the assignments $H_i = H_{i-1}$ and $E_i = E_{i-1} \cup \{e_i\}$, preserve the invariant.

After processing all the eaves in the manner stated above, we check whether the final hourglass collapses. If so, then there is no convex visibility segment. Otherwise, the cross tangents of the hourglass are convex visibility segments.

Analysis of the algorithm

We show that our algorithm runs in $O(n^2 \log n)$ time. The shortest path maps rooted at the vertices of P can be computed in a total of $O(n^2)$ time (see [8]). The maps can also be constructed from shortest path trees; these trees can be constructed in $O(n^2)$ time [7]. The extensions of all eaves can be constructed from the maps in $O(n^2)$ time.

It is a straightforward task to verify that Case 1 requires $O(n^2)$ running time. Determining whether each set E_i, $1 \leqslant i \leqslant r$, in Case 1 is empty, and computing E_i if it is not empty, requires $O(n)$ time (see Corollary 2.7). The total time for the computation of these sets is therefore $O(n^2)$.

As in the computation of each set E_i in Case 1, the computations of the sets E and F in Case 2 require $O(n)$ time. The overall running time of Case 2 is easily seen to be $O(n^2)$.

Now consider Case 3. An important step in the ith iteration in Case 3 is the location of the endpoints of the extension of e_i with respect to the corners ll, lr, ul and ur, of the current hourglass H_{i-1}. The corners of the hourglass H_i must either be vertices of P or endpoints of extensions of eaves of P. With each eave we store the labels of the vertices of the two edges on which the extension of the eave terminates. Moreover, for each edge v_iv_{i+1} of P, we store the eaves whose extensions terminate on v_iv_{i+1}, in sorted order along the edge. This sorting preprocessing clearly requires a total of $\mathrm{O}(n^2 \log n)$ time. (This step is indeed the prohibitive step in our algorithm, resulting in an $\mathrm{O}(n^2 \log n)$ bound on the running time.) Such an ordering of endpoints of eave extensions is used to identify eaves to be sent for reprocessing in cases (ii) and (iii); a simple linear scan of the portion(s) of the base of H_{i-1} that must be discarded, yields the modified base of H_i. Since the scan is performed only on portions to be discarded, and these portions of $\mathrm{bd}(P)$ are never scanned again in the algorithm, the total cost incurred in the steps for identifying eaves for reprocessing is $\mathrm{O}(n^2)$. The remaining steps in Case 3 are easily seen to be requiring $\mathrm{O}(n^2)$ time over all the $\mathrm{O}(n^2)$ iterations.

Now we show that an eave may require reprocessing only once. Only those eaves which have been processed earlier and found to be connecting both bases of the hourglass may be selected for reprocessing in cases (ii) and (iii). An eave selected for reprocessing, as mentioned above, does not meet one base of H_i and, therefore, will never meet both base of any hourglass in subsequent iterations. (In any iteration, the bases of the hourglass may shrink or remain unaltered but may never enlarge.) So, such an eave will never be sent for reprocessing again.

The final step of the algorithm involves computation of the two sides of the hourglass and the two cross tangents of the hourglass. This can easily be done in $\mathrm{O}(n)$ time. We summarize the result as follows.

Theorem 3.5. *Given a simple polygon P of n vertices, it is possible to compute a convex visibility segment (if one exists) in $\mathrm{O}(n^2 \log n)$ time and space.*

4. Conclusions

As stated in Theorem 2.3, a convex visibility segment must intersect the unforbidden region of each eave of the polygon. In order to compute the convex visibility segment, we need to compute the shortest paths between all pairs of vertices of the given polygon. A totally different approach may be needed to design an $\mathrm{o}(n^2 \log n)$ algorithm.

Theorem 2.3 also implies that the the intersection of the unforbidden regions of all eaves of the polygon is indeed the *palm kernel* of the polygon [6]; the entire polygon is convex visible from each point in the palm kernel. The computation of the palm kernel requires $\mathrm{O}(E)$ time [6], where E is the size of the visibility graph of the given polygon. No $\mathrm{o}(n^2)$ algorithm is known for computing the palm kernel of a polygon.

We feel that convex visibility may find applications in modeling reachability by multi-link robot arms where the links may bend at their joints either only in the left direction, or only in the right direction. We call such an arm, a *convex robot arm.* It may be convenient and desirable to place the robot at such a location that points in the workspace could be accessed by a convex robot arm. The existence of a non-empty palm kernel permits such robot location. If the palm kernel is empty then we may still reach each point of the polygonal workspace by a convex robot arm provided the workspace is weakly convex visible; we can move the robot along a convex visibility segment in order to reach an arbitrary point in the workspace.

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